

Assignment IV MTH 512 , Fall 2018

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QUESTION 1. Let < , > be the dot product on P_4. Given $W = Span\{x^3 + x + 1, x^3 + x\}$ is a subspace of P_4. Find the orthogonal complement of W in P_4 [Hint: Use the fake S.M.R of T and the Fake S.M.R of T^*, assume that the inner product is the dot product]

$W \approx W' = span\{(1,0,1,1), (1,0,1,0)\}$ subspace of $\mathbb{R}^4$

$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $\quad M^* = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\Rightarrow Range(T^*) \cong W'$

fake s.M.R of $T: \mathbb{R}^4 \to \mathbb{R}^2$ $\quad$ fake s.M.R of $T^*: \mathbb{R}^2 \to \mathbb{R}^4$

$Range(T^*)^\perp = Z(T)$

Solve homogenous: $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \overset{-R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$

$\overset{R_2+R_1 \to R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \Rightarrow x_4 = 0 \quad x_2$ free
$x_1 = -x_3$

$Z(T) = \{(-x_3, x_2, x_3, 0)\} = \{x_2(0,1,0,0) + x_3(-1,0,1,0)\}$
$= span\{(0,1,0,0), (-1,0,1,0)\} = Range(T^*)^\perp$

$\Rightarrow W^\perp = span\{x^2, -x^3 + x\}$

QUESTION 2. Let < , > be the dot product defined on R^2, and R^3. Given $T \in Hom_R(R^2, R^3)$ such that $T(a_1, a_2) = (2a_1 + a_2, -a_1 + 4a_2, -5a_2)$. Find $T^* \in Hom_R(R^3, R^2)$.

$\langle T(v), w \rangle = \langle v, T^*(w) \rangle \quad \forall\, v \in \mathbb{R}^2,\ w \in \mathbb{R}^3$

$\langle (2a_1 + a_2, -a_1 + 4a_2, -5a_2), (b_1, b_2, b_3) \rangle = \langle (a_1, a_2), T^*(b_1, b_2, b_3) \rangle$

$2a_1 b_1 + a_2 b_1 - a_1 b_2 + 4a_2 b_2 - 5a_2 b_3 = \langle (a_1, a_2), T^*(b_1, b_2, b_3) \rangle$

$a_1(2b_1 - b_2) + a_2(b_1 + 4b_2 - 5b_3) = \langle (a_1, a_2), T^*(b_1, b_2, b_3) \rangle$

$\Rightarrow T^*(b_1, b_2, b_3) = (2b_1 - b_2, b_1 + 4b_2 - 5b_3)$

QUESTION 3. Let $V = HOM_R(R^3, R^2) = L_R(R^3, R^2)$. Find $dim(V)$ and find a basis for V.

$dim(V) = 2 * 3 = 6 \qquad V \approx \mathbb{R}^{2\times 3} \approx \mathbb{R}^6$

Basis of $\mathbb{R}^{2\times3} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Basis of $V = \{T_1, T_2, T_3, T_4, T_5, T_6\}$ where $T_1, --, T_6 : \mathbb{R}^3 \to \mathbb{R}^2 \in Hom_R(\mathbb{R}^3, \mathbb{R}^2)$

$T_1(a_1, a_2, a_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (a_1, 0)$

$T_2(a_1, a_2, a_3) = (a_2, 0)$

$T_3(a_1, a_2, a_3) = (a_3, 0)$

$T_4(a_1, a_2, a_3) = (0, a_1)$

$T_5(a_1, a_2, a_3) = (0, a_2)$

$T_6(a_1, a_2, a_3) = (0, a_3)$

QUESTION 4. (Stare at the above question). Let $V = Hom_R(P_3, P_2)$. Find $dim(V)$ and find a basis for V.

$dim(V) = 6$

Basis of $V = \{T_1, T_2, T_3, T_4, T_5, T_6\}$ where $T_1, --, T_6 : P_3 \to P_2$

$T_1(a_1x^2 + a_2x + a_3) = a_1 x$

$T_2(a_1x^2 + a_2x + a_3) = a_2 x$

$T_3(a_1x^2 + a_2x + a_3) = a_3 x$

$T_4(a_1x^2 + a_2x + a_3) = a_1$

$T_5(a_1x^2 + a_2x + a_3) = a_2$

$T_6(a_1x^2 + a_2x + a_3) = a_3$

QUESTION 5. (short proof) Let $T_1 \in HOM_R(V, V), T_2 \in HOM_R(W, W)$, and let $X = V \oplus W$. Define a linear transformation $L : X \to X$ such that $L(v, w) = (T_1(v), T_2(w))$. We know (class notes) if a is an eigenvalue of T_1 or T_2, then a is an eigenvalue of L. Now prove the converse, i.e., Show that if c is an eigenvalue of L, then c is an eigenvalue of T_1 or T_2.

c is an eigenvalue of $L \Rightarrow \exists\, (v, w) \in V \oplus W$ & $(v, w) \neq (0_v, 0_w)$

s.t $L(v, w) = c(v, w)$

$L(v, w) = (T_1(v), T_2(w)) = (cv, cw) = c(v, w)$

$\Rightarrow T_1(v) = cv$ & $T_2(w) = cw$ where $v \neq 0_v$ & $w \neq 0_w$

$\Rightarrow c$ is an eigenvalue of T_1 & T_2

QUESTION 6. Let $B = span\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\}$. We know that $< A, B >= Trace(AB^T)$ is an inner product on $R^{2\times 2}$. Find an orthogonal basis for B (under $<,>$) Let $v_1 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ $v_2 = \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}$ $v_3 = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$

Orthogonal basis of B : w_1, w_2, w_3

$w_1 = v_1$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix} - \frac{Trace\left(\begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)}{Trace\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)} \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix} - \frac{Trace\left(\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}\right)}{Trace\left(\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}\right)} \cdot \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}$$

$w_1 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ $w_2 = \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}$

$$w_3 = v_3 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} - \frac{Trace\left(\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\1 & 0\end{bmatrix}\right)}{Trace\left(\begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\1 & 0\end{bmatrix}\right)}$$

$$* \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} - \frac{Trace\left(\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)}{4} \cdot \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} - \frac{Trace\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)}{Trace\left(\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\right)} \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} - \frac{Trace\left(\begin{bmatrix}2 & 2\\3 & 3\end{bmatrix}\right)}{4} \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} - \frac{0}{2} \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} - \frac{5}{4} \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} - \begin{bmatrix}5/4 & 5/4\\5/4 & 5/4\end{bmatrix} = \begin{bmatrix}-1/4 & -1/4\\-1/4 & 3/4\end{bmatrix}$$

$w_1 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ $w_2 = \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}$ $w_3 = \begin{bmatrix}-1/4 & -1/4\\-1/4 & 3/4\end{bmatrix}$

$B = Span\{w_1, w_2, w_3\}$ where w_1, w_2, w_3 are orthogonal under $\langle A, B\rangle = Trace(AB^T)$

If we decided to use the dot-product on $R^{2\times 2}$, will the elements of the basis that you calculated above stay orthogonal?

Yes, they will stay orthogonal

$$\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} = -1+1=0, \quad \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1/4\\-1/4\\-1/4\\3/4\end{bmatrix} = -\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{3}{4}=0$$

$$\begin{bmatrix}-1 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}-1/4\\-1/4\\-1/4\\3/4\end{bmatrix} = \longrightarrow \tfrac{1}{4}-\tfrac{1}{4}=0$$

QUESTION 7. (Short proof) Let W be a proper subspace of a finite dimensional vector space V.

(a) Show that there is a subspace L of V such that $V \approx W \oplus L$.(do not write much details)

$\dim(V) = n < \infty$, $\dim(W) = m < n$

Let $B = \{b_1, -- b_m\}$ be a basis of W,

Extend B to a basis of V, $V = span\{b_1, --, b_m, c_1, -- c_{n-m}\}$

Let $L = span\{c_1, --, c_{n-m}\}$, L is a subspace of V

for $v \in V$, $v = \alpha_1 b_1 + -- + \alpha_m b_m + \beta_1 c_1 + -- + \beta_{n-m} c_{n-m}$

Define $T: V \to W \oplus L$ $T(v) = (\alpha_1 b_1 + -- + \alpha_m b_m, \beta_1 c_1 + -- + \beta_{n-m} c_{n-m})$

It's clear that $T(v) = (0_v, 0_v)$ only if $v = 0_v \Rightarrow V \approx W \oplus L$

(b) Show that there is a subspace L of V such that $V = W + L$.

use the same L in (a)

For any $v \in V$, $v = \underbrace{\alpha_1 b_1 + -- + \alpha_m b_m} + \underbrace{\beta_1 c_1 + -- + \beta_{n-m} c_{n-m}} \Rightarrow v \in W + L$

For any $w \in W+L$, $w = m_1 + m_2$ s.t. $m_1 \in W, m_2 \in L$, $w = a_1 b_1 + -- + a_m b_m + d_1 c_1 + -- + d_{n-m} c_{n-m}$ $\Rightarrow w \in V$ $\Rightarrow V = W + L$

QUESTION 8. Let $V = span\{(1,1,1,1,1), (0,1,1,1,1), (-1,-1,0,0,0)\}$ and $W = \{(0,0,1,1,1), (1,2,2,2,2), (0,0,-1,-1,1)\}$.
Find a basis for $V + W$ and find a basis for $V \cap W$.

$$\begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} & \begin{matrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \end{matrix} \end{array} \quad \begin{matrix} v_1 + v_3 \to v_3 \\ \sim \\ -v_1 + w_2 \to w_2 \end{matrix} \quad \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} & \begin{matrix} v_1 \\ v_2 \\ v_1+v_3 \\ w_1 \\ -v_1+w_2 \\ w_3 \end{matrix} \end{array}$$

$-v_2 + (-v_1 + w_2) \to -v_1 + w_2$

$-(v_1 + v_3) + w_1 \to w_1$

$\sim$

$(v_1 + v_3) + w_3 \to w_3$

$$\begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} & \begin{matrix} v_1 \\ v_2 \\ v_1+v_3 \\ -v_1-v_3+w_1 \\ -v_2-v_1+w_2 \\ v_1+v_3+w_3 \end{matrix} \end{array}$$

$V + W = span\{(1,1,1,1,1), (0,1,1,1,1), (-1,-1,0,0,0), (0,0,-1,-1,1)\}$

$w_1 = v_1 + v_3 = (0,0,1,1,1)$ $\qquad w_2 = v_1 + v_2 = (1,2,2,2,2)$

$V \cap W = span\{(0,0,1,1,1), (1,2,2,2,2)\}$

QUESTION 9. Let $B = span\{x^2+1, x^2\}$ be a subspace of P_3. We know $< f(x), k(x) >= \int_0^1 f(x)k(x)\,dx$ is an inner product on P_3. Find an orthogonal basis for B (under $<\,,\,>$).

$V_1 = x^2+1 \quad V_2 = x^2 \quad , \quad W_1 = x^2+1$

$$W_2 = V_2 - \frac{\langle V_2, W_1\rangle}{\langle W_1, W_1\rangle} W_1 = x^2 - \frac{\langle x^2, x^2+1\rangle}{\langle x^2+1, x^2+1\rangle} x^2+1 = x^2 - \frac{\int_0^1 x^4+x^2\,dx}{\int_0^1 x^4+2x^2+1\,dx}(x^2+1$$

$$= x^2 - \frac{\left[\frac{x^5}{5}+\frac{x^3}{3}\right]_0^1}{\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_0^1}(x^2+1) = x^2 - \frac{\left[\frac{1}{5}+\frac{1}{3}\right]}{\left[\frac{1}{5}+\frac{2}{3}+1\right]}(x^2+1)$$

$$= x^2 - \frac{2}{7}x^2 - \frac{2}{7} = \frac{5}{7}x^2 - \frac{2}{7}$$

orthogonal basis for B under $\langle , \rangle$: $\left\{x^2+1, \frac{5}{7}x^2 - \frac{2}{7}\right\}$

$B = span\left\{x^2+1, \frac{5}{7}x^2 - \frac{2}{7}\right\}$

see back of the page for check

QUESTION 10. Let $V = span\{\overset{V_1}{(1,2,-1,0)}, \overset{V_2}{(-1,-1,1,1)}\}$ be a subspace of $X = R^4$.

(a) Find a subspace W of R^4 such that $X = V + W$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Extend}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$W = \{\overset{W_1}{(0,0,1,0)}, \overset{W_2}{(0,0,0,1)}\} \qquad V \cap W = \{(0,0,0,0)\}$

(b) Find $P_1 = Proj_V^X$, $P2_2 = Proj_W^X$, and then find the standard matrix representation for P_1, P_2, say M_1, M_2. (You may calculate Q^{-1}, on the back of this page, but not in the given space). **(NOTE that our definition of Poj_V^X here is a the identity linear transformation from R^4 to R^4 such that $P_1(v) = v$ if $v \in V$ and $P_1(v) = 0_v$ if $v \notin V$... There is another definition of $Proj_V^X$ involving $<,>$ on X, but this is not what I mean here (just as we did in class)**

$$Q = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} V_1 & V_2 & W_1 & W_2 \end{matrix} \end{array} \qquad Q^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

$P_1 : \mathbb{R}^4 \to \mathbb{R}^4 , \quad P_1(a_1V_1 + a_2V_2 + b_1W_1 + b_2W_2) = a_1V_1 + a_2V_2$

$P_2 : \mathbb{R}^4 \to \mathbb{R}^4 , \quad P_2(a_1V_1 + a_2V_2 + b_1W_1 + b_2W_2) = b_1W_1 + b_2W_2$

$$M_1Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow M_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}}_{Q^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

$$M_2Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

(c) Is it true that $M_1^2 = M_1$ and $M_2^2 = M_2$? Find $M_1 + M_2$? are you surprised? What is M_1M_2? $M_1 M_2 = 0$-matrix

Yes, P_1 & P_2 are idempotent $\quad M_1 + M_2 = I_4 \quad$ Expected as per class notes

QUESTION 11. (Application of spectral theorem) Let $M = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Let $T \in HOM_R(R^3, R^3)$ such that

$T(a_1, a_2, a_3) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

$\mathbb{R}^3$ E_1 E_3

(a) Convince me that $V = W + U$ for some invariant subspaces W, V of R^3.

$$\alpha I_3 - M = \begin{bmatrix} \alpha-3 & 0 & -2 \\ 0 & \alpha-3 & -2 \\ 0 & 0 & \alpha-1 \end{bmatrix} \qquad |\alpha I_3 - M| = (\alpha-3)\begin{vmatrix} \alpha-3 & -2 \\ 0 & \alpha-1 \end{vmatrix} = (\alpha-3)^2(\alpha-1) \quad \text{—(1)}$$

$$\alpha_2 = 3 \qquad (3I_3 - M) = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \alpha_1 = 1 \quad (I_3 - M) = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspaces are invariant subspaces

$x_3 = 0$ $\qquad -2x_1 - 2x_3 = 0 \Rightarrow x_1 = -x_3$, $x_2 = -x_3$

$\Rightarrow E_3 = \text{Span}\{(1,0,0), (0,1,0)\}$, $E_1 = \text{Span}\{(-1,-1,1)\}$ —(2)

By (1) & (2) : T is diagonalizable, $E_1 \cap E_2 = \{0_V\}$, $E_1, E_2 \subseteq \mathbb{R}^3$

we know from class that E_1 & E_2 are invariant subspaces of $\mathbb{R}^3$

since $\dim(E_1) = 1$, $\dim(E_3) = 2$, $\dim(E_1 \cap E_3) = 0 \Rightarrow \dim(E_1 + E_3) = 3$ & $E_1 + E_3$ subspace of $\mathbb{R}^3$ $\Rightarrow \mathbb{R}^3 = E_1 + E_3$

(b) Let P_1 be the projection of R^3 onto W and P_2 be a projection of R^3 onto U. (Note that each is idempotent, $P_1P_2 = 0$, and $P_1 + P_2 = I$ (the identity map)).

$W = E_1 = \text{Span}\{\underbrace{(-1,-1,1)}_{b_1}\}$ $\quad U = E_3 = \text{Span}\{\underbrace{(1,0,0)}_{b_2}, \underbrace{(0,1,0)}_{b_3}\}$ $\quad V = \mathbb{R}^3$

$P_1 : \mathbb{R}^3 \to \mathbb{R}^3$, $\quad P_2 : \mathbb{R}^3 \to \mathbb{R}^3$

$v \in \mathbb{R}^3 = a_1b_1 + a_2b_2 + a_3b_3$ $\qquad P_1(a_1b_1 + a_2b_2 + a_3b_3) = a_1b_1$ $\qquad a_1, a_2, a_3 \in \mathbb{R}$

$P_2(a_1b_1 + a_2b_2 + a_3b_3) = a_2b_2 + a_3b_3$

(c) Find the standard matrix representation of P_1 and P_2, say M_1, M_2. (Note that M_1, M_2 are idempotents, $M_1M_2 = 0$-matrix and $M_1 + M_2 = I_3$)

$$M_1 \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{L} \Rightarrow M_1 = LQ^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{M_1}$$

$$M_2 Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{M_2}$$

$M_1 + M_2 = I_3$ & $M_1M_2 = 0$-matrix

$\alpha_1 = 1$

d) Find the standard matrix representation of $P_1(T), P_2(T)$, say L_1, L_2. (See class notes, and (c) and you are done). (Note that $L_1 = aM_1$ and $L_2 = bM_2$, $L_1L_2 = 0$-matrix, Only one of them in this question is idempotent, and $L_1 + L_2 = M$, i.e. $P_1(T) + P_2(T) = T$.

$$L_1 = \alpha_1 M_1 = 1 \cdot M_1 = M_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad L_2 = \alpha_2 M_2 = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow L_1L_2 = 0\text{-matrix} \qquad L_1 + L_2 = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} = M \Rightarrow P_1(T) + P_2(T) = T$$

QUESTION 12. Let $A = \begin{bmatrix} 3 & 6 & 6 \\ -3 & -3 & -9 \\ 0 & 0 & 9 \end{bmatrix}$. Find the smith-form of A over Z, i.e., find invertible matrices R and C over Z such that $RAC = D$, where D is a diagonal matrix with d_1, d_2, d_3 are on the main diagonal such that $d_1 \mid d_2 \mid d_3$ and $d_1d_2d_3 = \pm|A|$.

$$A = \begin{bmatrix} 3 & 6 & 6 \\ -3 & -3 & -9 \\ 0 & 0 & 9 \end{bmatrix} \underset{\sim}{R_1 + R_2 \to R_2} \begin{bmatrix} 3 & 6 & 6 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix} = B \qquad |A| = |B| = 3 * 3 * 9 = 81$$

$$\Rightarrow d_1 | d_2 | d_3 = 81 \qquad d_1 = \pm 3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ -3 & -3 & -9 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{-2R_2 + R_1 \to R_1} \begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{-4C_1 + C_3 \to C_3}$$

$$\begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{C_1 + C_3 \to C_3} \underbrace{\begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R}\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{C}$$

$$d_1 * d_2 * d_3 = 3 * 3 * 9 = 81 = |A|$$

$$\begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ -3 & -3 & -9 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \Rightarrow RAC = D$$

$|R| = 1 \quad |C| = 1$

$\Rightarrow R$ & C invertible over Z

QUESTION 13. Define a function $<,>$ on R^2 such that $< (a_1, a_2), (b_1, b_2) >= |a_1b_1| + |a_2b_2|$. Convince me that $<,>$ is NOT an inner product on R^2.

$$<(a_1, a_2) + (c_1, c_2), (b_1, b_2)> = <(a_1 + c_1, a_2 + c_2), (b_1, b_2)>$$

$$= |(a_1 + c_1)b_1| + |(a_2 + c_2)b_2|$$

$$= |a_1b_1 + c_1b_1| + |a_2b_2 + c_2b_2| \le |a_1b_1| + |c_1b_1| + |a_2b_2| + |c_2b_2|$$

$$<(a_1, a_2), (b_1, b_2)> + <(c_1, c_2), (b_1, b_2)> = |a_1b_1| + |a_2b_2| + |c_1b_1| + |c_2b_2|$$

$$\Rightarrow <(a_1, a_2) + (c_1, c_2), (b_1, b_2)> \le <(a_1, a_2), (b_1, b_2)> + <(c_1, c_2), (b_1, b_2)>$$

(violates the axiom)

must be equal

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Assignment IV MTH 512 , Fall 2018

Ayman Badawi

QUESTION 1. Let $<,>$ be the dot product on P_4. Given $W = Span\{\overset{w_1}{x^3+x+1}, \overset{w_2}{x^3+x}\}$ is a subspace of P_4. Find the orthogonal complement of W in P_4 [Hint: Use the fake S.M.R of T and the Fake S.M.R of T^*, assume that the inner product is the dot product]

define $T: P_4 \to P_2$

let M be the fake S.M.R of T:

$T(ax^3+bx^2+cx+d) = M\cdot\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$

$M = \begin{bmatrix}1&0&1&1\\1&0&1&0\end{bmatrix}$

define $T^*: P_2 \to P_4$

let M^* be the fake S.M.R of T^*:

$T^*(a_1w_1+a_2w_2) = M^*\begin{bmatrix}a_1\\a_2\end{bmatrix}$

$M^* = (\bar{M})^T = \begin{bmatrix}1&1\\0&0\\1&1\\1&0\end{bmatrix}$

we know: $Range(T^*)^\perp = Z(T) \Rightarrow W^\perp = Z(T)$

$Z(T) \Rightarrow \begin{bmatrix}1&0&1&1\\1&0&1&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

- $a+c+d=0 \Rightarrow -c+c+\boxed{d=0}$
- $a+c=0 \Rightarrow a=-c$

$W^\perp \Rightarrow \left(\begin{bmatrix}-c\\b\\c\\0\end{bmatrix}\right) = +c\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$W^\perp = Span\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$

in P_4:

$W^\perp = Span\{(-x^3+x), x^2\}$

and $<w_i, w_j^\perp> = 0$

QUESTION 2. Let $<,>$ be the dot product defined on R^2, and R^3. Given $T \in Hom_R(R^2, R^3)$ such that $T(a_1, a_2) = (2a_1 + a_2, -a_1 + 4a_2, -5a_2)$. Find $T^* \in Hom_R(R^3, R^2)$.

$T: R^2 \to R3$

$* \; T(a_1, a_2) = (2a_1+a_2, -a_1+4a_2, -5a_2)$

let M be the S.M.R of T:

$M = \begin{bmatrix}2&1\\-1&4\\0&-5\end{bmatrix}$

let $T^*: R^3 \to R^2$

$T^*(b_1, b_2, b_3) = (\text{—}, \text{—})$

let M^* be the S.M.R of T^*.

$M^* = \begin{bmatrix}2&-1&0\\1&4&-5\end{bmatrix}$

$T^*(b_1, b_2, b_3) = M^*\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

$= \begin{bmatrix}2&-1&0\\1&4&-5\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

$= b_1\begin{bmatrix}2\\1\end{bmatrix} + b_2\begin{bmatrix}-1\\4\end{bmatrix} + b_3\begin{bmatrix}0\\-5\end{bmatrix}$

$T^*(b_1, b_2, b_3) = (2b_1-b_2, b_1+4b_2-5b_3)$

$T^* \in Hom(R^3, R_2)$

QUESTION 3. Let $V = HOM_R(R^3, R^2) = L_R(R^3, R^2)$. Find $dim(V)$ and find a basis for V.

we know $HOM_R(R^3, R^2) \approx \mathbb{R}^6$;

So $\dim(V) = \dim(R^6) = 6$

we need to find 6 indp linear transf. from $R^3 \to R^2$

- $T_1 : R^3 \to R^2$
  $T_1(a,b,c) = (a,0)$
- $T_2 : R^3 \to R_2$
  $T_2(a,b,c) = (b,0)$
- $T_3 : R^3 \to R^2$
  $T_3(a,b,c) = (c,0)$

$T_4 : R^3 \to R^2$
$T_4(a,b,c) = (0,a)$

- $T_5 : R^3 \to R^2$
  $T_5(a,b,c) = (0,b)$
- $T_6 : R^3 \to R^2$
  $T_6(a,b,c) = (0,c)$

Basis for $V = \text{Span}\{T_1, T_2, T_3, T_4, T_5, T_6\}$

QUESTION 4. (Stare at the above question). Let $V = Hom_R(P_3, P_2)$. Find $dim(V)$ and find a basis for V.

$Hom_R(P_3, P_2) \approx P^6 \cong \mathbb{R}^6$

$\dim(V) = 6$

$T_1 : P_3 \to P_2$

$T_1(ax^2+bx+c) = ax$

- $T_2 : P_3 \to P_2$
  $T_2(ax^2+bx+c) = bx$
- $T_3 : P_3 \to P_2$
  $T_3(ax^2+bx+c) = cx$

- $T_4 : P_3 \to P_2$
  $T_4(ax^2+bx+c) = a$
- $T_5 : P_3 \to P_2$
  $T_5(ax^2+bx+c) = b$
- $T_6 : P_3 \to P_2$
  $T_6(ax^2+bx+c) = c$

Basis for $V = \text{span}\{T_1, T_2, T_3, T_4, T_5, T_6\}$.

QUESTION 5. (short proof) Let $T_1 \in HOM_R(V,V), T_2 \in HOM_R(W,W)$, and let $X = V \oplus W$. Define a linear transformation $L : X \to X$ such that $L(v,w) = (T_1(v), T_2(w))$. We know (class notes) if a is an eigenvalue of T_1 or T_2, then a is an eigenvalue of L. Now prove the converse, i.e., Show that if c is an eigenvalue of L, then c is an eigenvalue of T_1 or T_2.

$\to$ take c_1 be the eigenvalue of L corresponding to the eigen space $(v,0) \in X$

$L(v,0) = c_1(v,0)$

$\therefore L(v,0) = (T_1(v), T_2(0)) = c_1(v,0) = (c_1 v, 0) \Rightarrow T_1(v) = c_1 v \therefore c_1$ is an eigenvalue of T_1

$\to$ take c_2 be the eigenvalue of L corresponding to the eigenspace $(0,w) \in X$

$L(0,w) = (T_1(0), T_2(w)) = c_2(0,w) = (0, c_2 w)$. $\therefore c_2$ is an eigenvalue of T_2

$\to$ Take c_3 be the eigenvalue of L for $(v,w) \in X$.

$L(v,w) = c_3(v,w) = (c_3 v, c_3 w) = (T_1(v), T_2(w)) \Rightarrow \left. \begin{array}{l} T_1(v) = c_3 v \\ T_2(w) = c_3 w \end{array} \right\}$ c_3 eigenvalue for both T_1, T_2

Therfore, if c is an eigenvalue of L then c is an eigenvalue of T_1 or T_1 or T_1 & T_2.

QUESTION 6. Let $B = span\{\overset{V_1}{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}}, \overset{V_2}{\begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}}, \overset{V_3}{\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}}\}$. We know that $< A, B >= Trace(AB^T)$ is an inner product on $R^{2\times2}$. Find an orthogonal basis for B (under $<,>$)

→ using Gram Schmidt:

let $B^\perp = \text{Span}\{w_1, w_2, w_3\}$

$* \; w_1 = v_1 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$* \; w_2 = v_2 - \dfrac{\langle v_2, w_1\rangle w_1}{\langle w_1, w_1\rangle}$

$= \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix} - \dfrac{4}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}$

$* \; w_3 = v_3 - \cancel{\dfrac{\overset{0}{\langle v_3, w_2\rangle} w_2}{\langle w_2, w_2\rangle}} - \dfrac{\langle v_3, w_1\rangle w_1}{\langle w_1, w_1\rangle}$

$= \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} - \dfrac{5}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}-1/4 & -1/4\\-1/4 & 3/4\end{bmatrix}$

So $B^\perp = \text{span}\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}-1/4 & -1/4\\-1/4 & 3/4\end{bmatrix}\}$

But if w_3 is orthogonal so is $4w_3$

$4w_3 = \begin{bmatrix}-1 & -1\\-1 & 3\end{bmatrix}$

So $\underline{B}^\perp = \text{Span}\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}-1 & -1\\-1 & 3\end{bmatrix}\}$

Calculations

$\Rightarrow \langle v_2, w_1\rangle = \text{trace}(v_2 w_1^T) = \text{trace}(\begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix})$

$= \text{trace}(\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}) = 4$

$\Rightarrow \langle w_1, w_1\rangle = \text{trace}(w_1 w_1^T) = \text{trace}(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix})$

$= \text{trace}(\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}) = 4$

$\Rightarrow \langle v_3, w_2\rangle = \text{trace}(v_3 w_2^T) = \text{trace}(\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\1 & 0\end{bmatrix})$

$= \text{trace}(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}) = 0$

$\therefore \langle v_3, w_2\rangle = 0$

$\Rightarrow \langle v_3, w_1\rangle = \text{trace}(v_3 w_1^T) = \text{trace}(\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix})$

$= \text{trace}(\begin{bmatrix}2 & 2\\3 & 3\end{bmatrix}) = 5$

If we decided to use the dot-product on $R^{2\times2}$, will the elements of the basis that you calculated above stay orthogonal?

Yes because we still get $\langle w_i, w_j\rangle = 0$ $i \neq j$. $1 \leq i \leq 3$, $1 \leq j \leq 3$ with using the dot product as the inner product.

QUESTION 7. (Short proof) Let W be a proper subspace of a finite dimensional vector space V.

(a) Show that there is a subspace L of V such that $V \approx W \oplus L$.(do not write much details) $\underline{*Z(T) = 0_V}$.

To show isomorphism we show $\dim(V) = \dim(W \oplus L)$ and define $T: V \to W \oplus L$ s.t. $Z(T) = 0_V$

let $\dim(V) = n < \infty$

take $\dim(W) = k < n$ & $\dim(L) = n - k$

then $\dim(V) = \dim(W \oplus L)$

$n = n - k + k$

take
- $V = \text{span}\{b_1, b_2, \dots, b_n\}$
- $W = \text{span}\{w_1, w_2, \dots, w_k\}$
- $L = \text{span}\{\ell_1, \ell_2, \dots, \ell_{n-k}\}$

$\Big\}$ basis

take $v \in Z(T) \Rightarrow v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

$T(v) = (d_1 w_1 + \dots + d_k w_k, c_1 \ell_1 + \dots + c_{n-k} \ell_{n-k}) = (0, 0)$

$d_1 w_1 + \dots + d_k w_k = 0$ iff $d_i = 0$
$c_1 \ell_1 + \dots + c_{n-k} \ell_{n-k} = 0$ iff $c_i = 0$ $\Rightarrow$ $\underline{Z(T) = 0_V}$

(b) Show that there is a subspace L of V such that $V = W + L$.

we know if L, W are subspaces of V then.

$L + W = \{\ell + w \mid \ell \in L, w \in W\}$ is a subspace of V

But $\dim(L + W) = n - k + k + \dim(L \cap W)$

from definition

taking $L \cap W = 0$ $\dim(L + W) = n$

hence we can find n linearly. Indp. elements to span $(L + W)$ hence $\text{span}\{(L+W)\} = \text{span}\{V\}$

$L + W = V$

QUESTION 8. Let $V = span\{(1,1,1,1,1), (0,1,1,1,1), (-1,-1,0,0,0)\}$ and $W = \{(0,0,1,1,1), (1,2,2,2,2), (0,0,-1,-1,1)\}$.
Find a basis for $V + W$ and find a basis for $V \cap W$.

let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{c} v_1 \\ v_2 \\ v_1+v_3 \\ w_1 \\ w_2 - v_1 \\ w_3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{c} v_1 \\ v_2 \\ v_1+v_3 \\ -(v_1+v_3)+w_1 \\ w_2 - v_1 - v_2 \\ v_1+v_3+w_3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} \} \in V + W \\ \\ \} \in V \cap W \\ \\ \} \in V + W \end{array}$

the Basis for $V + W = \text{span}\{(1,1,1,1,1), (0,1,1,1,1), (0,0,0,0,2)\}$

or $\text{span}\{(1,1,1,1,1), (0,1,1,1,1), (0,0,-1,-1,1)\}$

the Basis for $V \cap W$ is found from the zero rows;

$-(v_1 + v_3) + w_1 = (0,0,0,0,0)$

$w_1 = v_1 + v_3 = (0,0,1,1,1)$

$w_2 - v_1 - v_2 = (0,0,0,0,0)$

$w_2 = (v_1 + v_2) = (1,2,2,2,2)$

The Basis for $V \cap W = \text{span}\{(0,0,1,1,1), (1,2,2,2,2)\}$

QUESTION 9. Let $B = span\{\overset{v_1}{x^2+1}, \overset{v_2}{x^2}\}$ be a subspace of P_3. We know $< f(x), k(x) >= \int_0^1 f(x)k(x)\,dx$ is an inner product on P_3. Find an orthogonal basis for B (under $<\,,\,>$).

using Gram-Schmidt :

let $B^\perp = span\{w_1, w_2\}$ be the orthogonal basis for B.

$w_1 = v_1 = x^2+1$

$w_2 = v_2 - \dfrac{\langle v_2, w_1\rangle w_1}{\langle w_1, w_1\rangle}$

$= x^2 - \dfrac{\langle x^2, x^2+1\rangle (x^2+1)}{\langle x^2+1, x^2+1\rangle}$

- $\langle x^2, x^2+1\rangle = \int_0^1 x^4 + x^2\,dx = \dfrac{x^5}{5} + \dfrac{x^3}{3}\Big|_0^1 = \dfrac{8}{15}$

- $\langle x^2+1, x^2+1\rangle = \int_0^1 x^4 + 2x^2 + 1\,dx = \dfrac{x^5}{5} + \frac{2}{3}x^3 + x\Big|_0^1$
$= \dfrac{28}{15}$

$w_2 = x^2 - \dfrac{8/15}{28/15}(x^2+1)$

$= x^2 - \frac{2}{7}x^2 - 2/7 = \frac{5}{7}x^2 - \frac{2}{7}$

but if w_2 orthogonal so is $\alpha w_2 \Rightarrow \alpha = \frac{7}{5}$

$\alpha w_2 = x^2 - 2/5$

$B^\perp = span\{(x^2+1), (x^2 - 2/5)\}$

QUESTION 10. Let $V = span\{\overset{v_1}{(1,2,-1,0)}, \overset{v_2}{(-1,-1,1,1)}\}$ be a subspace of $X = R^4$.
(a) Find a subspace W of R^4 such that $X = V + W$.

Find two more independent elements in $R^4 \setminus V$

$W = span\{\overset{w_1}{(0,0,1,0)}, \overset{w_2}{(0,0,0,1)}\}$

Sketch of finding W $= \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ leading

(b) Find $P_1 = Proj_V^X$, $P_2 = Proj_W^X$, and then find the standard matrix representation for P_1, P_2, say M_1, M_2. (You may calculate Q^{-1}, on the back of this page, but not in the given space). **(NOTE that our definition of Poj_V^X here is a the identity linear transformation from R^4 to R^4 such that $P_1(v) = v$ if $v \in V$ and $P_1(v) = 0_v$ if $v \notin V$... There is another definition of $Proj_V^X$ involving $<,>$ on X, but this is not what I mean here (just as we did in class)**

$P_1 : R^4 \to R^4$ (s.t range$(P_1) = V$)

$P_1(av_1 + bv_2 + cw_1 + dw_2) = av_1 + bv_2$

so $P_1(v_1) = v_1$, $P_1(v_2) = v_2$ where v_1, v_2 basis for V

$P_1(w_1) = 0_X$, $P_1(w_2) = 0_X$ where w_1, w_2 basis for W

$$[T]_{(v,e)} = \begin{array}{c} \begin{array}{cccc} P_1(v_1) & P_1(v_2) & P_1(w_1) & P_1(w_2) \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} = M_1 \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{Q}$$

$$M_1 = M_{1ve} Q^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

$P_2 : R^4 \to R^4$ (s.t range$(P_2) = W$)

$P_2(av_1 + bv_2 + cw_1 + dw_2) = cw_1 + dw_2$

so $P_2(v_1) = 0_X$, $P_2(v_2) = 0_X$

$P_2(w_1) = w_1$, $P_2(w_2) = w_2$

$$M_{2we} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M_2 \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{Q}$$

$$M_2 = M_{2we} Q^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

(c) Is it true that $M_1^2 = M_1$ and $M_2^2 = M_2$? Find $M_1 + M_2$? are you surprised? What is M_1M_2?

yes

$$M_1 + M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_1M_2 = 0_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T(v) \in V \ \forall v \in V$

QUESTION 11. (Application of spectral theorem) Let $M = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Let $T \in HOM_R(R^3, R^3)$ such that

$T(a_1, a_2, a_3) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. $M \Rightarrow$ diagonal.

(a) Convince me that $V = W + U$ for some invariant subspaces W, V of R^3.

$$|\lambda I - M| = \begin{vmatrix} \lambda-3 & 0 & -2 \\ 0 & \lambda-3 & -2 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-3)^2(\lambda-1) = 0$$

$\lambda_1 = 1, \ \lambda_{2,3} = 3$

the eigenspace for $\lambda = 1$, E_1

$$\begin{bmatrix} -3 & 0 & -2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2x_1 - 2x_3 = 0 \Rightarrow x_1 = -x_3 \\ -2x_2 - 2x_3 = 0 \Rightarrow x_2 = -x_3 \end{matrix}$$

$E_1 = \text{span}: \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$; $\boxed{\text{let } W = E_1}$

the eigen space corresponding to $\lambda = 3$, E_3

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_3 = 0, \ x_2, x_1 \text{ free}$$

$E_3 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\boxed{\text{let } U = E_3}$

So since the characteristic eq of the transfo. could be written as multiplication of linear factors and $\dim(E_{\alpha_i})$ = multiplicity of the corresponding eigenvalue

then T is diagonlizble.

and by the spectral theorem

$V = W + U$

where W and U are invarient subspaces

$\Rightarrow W = E_1$, take $w \in E_1$

$T(w) = 1 \cdot w \in E_1$

$\Rightarrow U = E_3$; take $u \in E_3$

$T(u) = 3 \cdot u \in E_3$

(b) Let P_1 be the projection of R^3 onto W and P_2 be a projection of R^3 onto U. (Note that each is idempotent, $P_1 P_2 = 0$, and $P_1 + P_2 = I$ (the identity map)).

$P_1 : R^3 \longrightarrow R^3$ (S.t Range$(P_1) = W$)

$P_1(aw_1 + bu_1 + cu_2) = I(aw_1) = aw_1$

where w_1 is a basis of W and u_1, u_2 are basis of U

$P_1(w_1) = w_1 = (-1, -1, 1)$

$P_1(u_1) = 0_V$; $P_1(u_2) = 0_V$

$P_2 : R^3 \longrightarrow R^3$ (S.t Range$(P_2) = U$)

$P_2(aw_1 + bu_1 + cu_2) = bu_1 + cu_2$

$P_2(w_1) = 0_V$,

$P_2(u_1) = u_1 = (1, 0, 0)$

$P_2(u_2) = u_2 = (0, 1, 0)$.

(c) Find the standard matrix representation of P_1 and P_2, say M_1, M_2. (Note that M_1, M_2 are idempotents, $M_1 M_2 = 0$-matrix and $M_1 + M_2 = I_3$)

$$1_{,W,e} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = M_1 \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q}$$

$$M_1 = M_{1We} Q^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longleftarrow$ calculations

$$M_{2Ue} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = M_2 \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q}$$

$$M_2 = M_{2Ue} Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where $M_1 + M_2 = I$

and $M_1 M_2 = 0$

d) Find the standard matrix representation of $P_1(T), P_2(T)$, say L_1, L_2. (See class notes, and (c) and you are done). (Note that $L_1 = aM_1$ and $L_2 = bM_2$, $L_1L_2 = $ 0-matrix, Only one of them in this question is idempotent, and $L_1 + L_2 = M$, i.e. $P_1(T) + P_2(T) = T$.

$L_1 = 1 \cdot M_1$

$L_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore L_1$ is idempotent since the eigenvalue $\alpha = 1$ so, $L_1^2(v) = L_1 = \alpha^2 v = v$

$L_2 = 3M_2$

$= 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

QUESTION 12. Let $A = \begin{bmatrix} 3 & 6 & 6 \\ -3 & -3 & -9 \\ 0 & 0 & 9 \end{bmatrix}$. Find the smith-form of A over Z, i.e., find invertible matrices R and C over Z such that $RAC = D$, where D is a diagonal matrix with d_1, d_2, d_3 are on the main diagonal such that $d_1 \mid d_2 \mid d_3$ and $d_1d_2d_3 = \pm|A|$. the gcd = 3 , $|A| = 81$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ -3 & -3 & -9 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 & 6 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-3R_2 + R_1 \to R_1$

$$\begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-4C_1 + C_3 \to C_3$

$$\begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 + C_3 \to C_3$

$$\underbrace{\begin{bmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \cdot \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{C}$$

QUESTION 13. Define a function $<,>$ on R^2 such that $<(a_1, a_2), (b_1, b_2)> = |a_1b_1| + |a_2b_2|$. Convince me that $<,>$ is NOT an inner product on R^2.

The properties of the inner product that fails are: take $\alpha \in F = R$

① $\langle \alpha(a_1, a_2), (b_1, b_2)\rangle = \langle(\alpha a_1, \alpha a_2), (b_1, b_2)\rangle$

$= |\alpha a_1 b_1| + |\alpha a_2 b_2| = |\alpha|(|a_1 b_1| + |a_2 b_2|)$

$= |\alpha|\langle(a_1, a_2), (b_1, b_2)\rangle$

since $\alpha \in R$ could be negative. $\neq \alpha\langle(a_1, a_2), (b_1, b_2)\rangle$

② $\langle(a_1, a_2) + (c_1, c_2), (b_1, b_2)\rangle = \langle(a_1 + c_1, a_2 + c_2), (b_1, b_2)\rangle$

$= |(a_1 + c_1)b_1| + |(a_2 + c_2)b_2| = |a_1 b_1 + c_1 b_1| + |a_2 b_2 + c_2 b_2|$

using tringle inequality:

$\leq |a_1 b_1| + |c_1 b_1| + |a_2 b_2| + |c_2 b_2|$

$= \langle(a_1, a_2), (b_1, b_2)\rangle + \langle(c_1, c_2), (b_1, b_2)\rangle$

So $\langle(a_1, a_2) + (c_1, c_2), (b_1, b_2)\rangle \leq \langle(a_1, a_2), (b_1, b_2)\rangle + \langle(c_1, c_2), (b_1, b_2)\rangle$

$\therefore$ hence the property fails.

So it's not an inner product space.

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